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ANALYTIC SOLUTION IN COMPLEX FORM FOR A CLASS OF ELASTIC MODULI FOR A TWO-DIMENSIONAL INHOMOGENEITY OF A BODY*

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A general solution is constructed for the two-dimensional problem of elasticity theory in the particular case of an inhomogeneity governed by the relationships

$$E = E_0 r^a \exp(b\theta); \quad \nu + 1 = r^a [c_1 + c_2 r \sin(\theta + c_3)] \exp(b\theta)$$

where E is Young's modulus, and ν is the Poisson's ratio. The constants c_1, c_2, c_3, a and b can be arranged so that the known condition $0 \leqslant \nu \leqslant {}^1/_2$ is satisfied in the closed domain of a truncated wedge. Specific boundary value problems are presented.

1. Basic equations. For a two-dimensional inhomogeneity of an isotropic body, the plane state of stress for a stress function $U(r, \theta)$, *no mass forces and temperature expansion*, is determined in a polar coordinate system r, θ by a linear differential equation with variable coefficients /1,2/

$$\Delta(e\Delta U) = \frac{1}{r}\Big[\left(\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2}\right)\frac{\partial^2 U}{\partial r^2} + \frac{\partial^2 f}{\partial r^2}\left(\frac{\partial U}{\partial r} + \frac{1}{r}\frac{\partial^2 U}{\partial \theta^2}\right) - \frac{2}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial r} - \frac{f}{r}\right)\frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial r} - \frac{U}{r}\right)\Big] \tag{1.1}$$

$$e = \frac{1}{E}, \quad f = \frac{1+\nu}{E}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

with appropriate boundary conditions.

The stress components are determined from the formulas

$$\sigma_r = \frac{1}{r}\left(\frac{\partial U}{\partial r} + \frac{1}{r}\frac{\partial^2 U}{\partial \theta^2}\right), \quad \tau = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial \theta}\right), \quad \sigma_\theta = \frac{\partial^2 U}{\partial r^2} \tag{1.2}$$

A number of boundary value problems has been examined earlier under the assumption that the Poisson's ratio is a constant and the Young's modulus is a function of the coordinates r, θ /1,2/. In particular, problems have been examined /2/ under the assumptions that the stress function U and the Young's modulus have the form (n, k, E_0 are constants)

$$U = r^n F(\theta), \quad E = E_0 r^{-k}\psi(\theta)$$

In this case, the problem for the function $F(\theta)$ reduces to appropriate boundary conditions and a fourth-order ordinary differential equation with variable coefficients dependent on a given function $\psi(\theta)$.

In contrast to /1–4/, a class of functions $E(r, \theta)$ and $\nu(r, \theta)$ is presented below, for whose stress functions a general solution is constructed in complex form and containing two arbitrary analytic functions of the complex variable $z = \ln r + i\theta$.

We henceforth examine the case when the right side in (1.1) is zero, i.e., the following conditions are satisfied for the function f

$$\frac{\partial^2 f}{\partial r^2} = 0, \quad \frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2} = 0, \quad \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial r} - \frac{f}{r}\right) = 0$$

which reduce after integration to the form

$$f = c_1 + c_2 r \sin(\theta + c_3), \quad \nu = fE - 1 \tag{1.3}$$

It is known that $0 < \nu \leqslant {}^1/_2$ for all materials /1–3/. In this case, by taking account of the expression for f in (1.1), we obtain the inequality

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$$1 < fE \leqslant 3/2 \tag{1.4}$$

As will be shown below, in problems with a bounded domain $r_1 \leqslant r \leqslant r_2$, $\theta_1 \leqslant \theta \leqslant \theta_2$, the constants c_1, c_2, c_3 and the constants in E can be selected so that the inequality (1.4) would be satisfied. The dependence (1.3) for ν agrees with the known formula of the experimental theory of elasticity /5/ in which the shear modulus equals $1/(2f)$.

Taking account of (1.3), we write (1.1) in the expanded form

$$e\Delta\Delta U + (\Delta e)(\Delta U) + 2\left(\frac{\partial e}{\partial r}\frac{\partial}{\partial r}\Delta U + \frac{1}{r^2}\frac{\partial e}{\partial \theta}\frac{\partial}{\partial \theta}\Delta U\right) = 0 \tag{1.5}$$

We extract that class of functions $e(r, \theta)$ for which the stress function satisfying (1.5) has the form

$$U = \alpha(r)\beta(\theta)\chi(z) + \varphi(z) \tag{1.6}$$

Here $\alpha(r)$, $\beta(\theta)$ are real functions of their arguments to be determined from (1.5), while $\chi(z)$, $\varphi(z)$ are arbitrary analytic functions of the logarithmic affix $z = \ln r + i\theta$.

2. Method of solution. We form the appropriate derivatives of (1.6) and we insert them into (1.5). We consequently obtain

$$f_1\chi + (f_2 + if_3)\chi' + (f_4 + if_5)\chi'' = 0 \tag{2.1}$$

$$f_1 = \left(\alpha''\beta + \frac{\alpha'\beta}{r} + \frac{\alpha\beta''}{r^2}\right)\Delta e + \frac{2}{r^2}\left(\alpha''\beta' + \frac{\alpha'\beta'}{r} + \frac{\alpha\beta'''}{r^2}\right)\frac{\partial e}{\partial\theta} + 2\left(\alpha'''\beta + \frac{\alpha''\beta}{r} - \frac{\alpha'\beta}{r^2} + \frac{\alpha'\beta''}{r^2} - \frac{2\alpha\beta''}{r^3}\right)\frac{\partial e}{\partial r} + \tag{2.2}$$

$$e\left(\alpha''''\beta + \frac{2\alpha'''\beta}{r} - \frac{\alpha''\beta}{r^2} + \frac{\alpha'\beta}{r^3} + \frac{2\alpha''\beta''}{r^2} - \frac{2\alpha'\beta''}{r^3} + \frac{4\alpha\beta''}{r^4} + \frac{\alpha\beta''''}{r^4}\right)$$

$$f_2 = \frac{2}{r}\left[2e\left(\alpha'''\beta + \frac{\alpha'\beta''}{r^2} - \frac{\alpha\beta''}{r^3}\right) + \alpha'\beta\Delta e + \left(3\alpha''\beta - \frac{\alpha'\beta}{r} + \frac{\alpha\beta''}{r^2}\right)\frac{\partial e}{\partial r} + \frac{2\alpha'\beta'}{r^2}\frac{\partial e}{\partial\theta}\right]$$

$$f_3 = \frac{2}{r^2}\left[2e\left(\alpha''\beta' - \frac{\alpha'\beta'}{r} + \frac{2\alpha\beta'}{r^2} + \frac{\alpha\beta'''}{r^2}\right) + \alpha\beta'\Delta e + \frac{2}{r}\left(\alpha'\beta' - \frac{2\alpha\beta'}{r}\right)\frac{\partial e}{\partial r} + \left(\alpha''\beta + \frac{\alpha'\beta}{r} + \frac{3\alpha\beta''}{r^2}\right)\frac{\partial e}{\partial\theta}\right]$$

$$f_4 = \frac{4}{r^2}\left[\left(\alpha''\beta - \frac{\alpha'\beta}{r} - \frac{\alpha\beta''}{r^2}\right)e + \alpha'\beta\frac{\partial e}{\partial r} - \frac{\alpha\beta'}{r^2}\frac{\partial e}{\partial\theta}\right]$$

$$f_5 = \frac{4}{r^3}\left[2\beta'\left(\alpha' - \frac{\alpha}{r}\right)e + \alpha\beta'\frac{\partial e}{\partial r} + \alpha'\beta\frac{\partial e}{\partial\theta}\right]$$

We now consider the Young's modulus (or equivalently, the function e) to be a function with separable variables

$$e = e_1(r)e_2(\theta), \quad e_1 = \frac{1}{E_1(r)}, \quad e_2 = \frac{1}{E_2(\theta)} \tag{2.3}$$

In this case, we note that if the coefficients f_4 and f_5 in (2.2) are equated to zero, we then arrive at two equations with separable variables which we write in the form

$$r^2\left(\frac{e_1'\alpha'}{e_1\alpha} + \frac{\alpha''}{\alpha} - \frac{\alpha'}{r\alpha}\right) = \frac{1}{\beta}\left(\frac{e_2'\beta'}{e_2} + \beta''\right) = p^2 = \text{const} \tag{2.4}$$

$$\frac{\alpha}{\alpha'}\left(\frac{e_1'}{e_1} + \frac{2\alpha'}{\alpha} - \frac{2}{r}\right) = -\frac{e_2'\beta}{e_2\beta'} = q^2 = \text{const}$$

Let us examine the case $p = 0$, $q = 1$. Then the system (2.4) becomes

$$\frac{e_1'}{e_1} + \frac{\alpha''}{\alpha'} - \frac{1}{r} = 0, \quad \frac{e_1'}{e_1} + \frac{\alpha'}{\alpha} - \frac{2}{r} = 0, \quad \frac{e_2'}{e_2} + \frac{\beta''}{\beta'} = 0, \quad \frac{e_2'}{e_2} + \frac{\beta'}{\beta} = 0 \tag{2.5}$$

Integrating the system of the first two and the last two equations in (2.5) and going over to the functions E and $\alpha\beta$ we obtain

$$E = E_0 r^a \exp(b\theta), \quad \alpha\beta = Ar^{a+2}\exp(b\theta) \tag{2.6}$$

It can be seen by direct substitution of (2.3), (2.6) into (2.2) that the coefficients f_1, f_2, f_3 vanish. Consequently, the function U in (1.6) takes the form

$$U = Ar^{a+2}\chi(z)\exp(b\theta) + \varphi(z) \tag{2.7}$$

Here a, b, E_0 are arbitrary real constants, and A is a complex constant.

If E is inserted by means of (2.6) into the initial equation (1.5), then in contrast to the case examined in /3/, we arrive at an equation with variable coefficients

$$r^2\Delta\Delta U + (a^2 + b^2)\Delta U - 2\left(ar\frac{\partial}{\partial r}\Delta U + b\frac{\partial}{\partial\theta}\Delta U\right) = 0 \tag{2.8}$$

The real and imaginary parts of the function U in (2.7), and their linear combinations, are real solutions of the linear equation (2.8) of (1.5).

Taking account of (2.7), we write (1.2) and (1.3) in the form

$$\nu + 1 = [c_1 + c_2 r\sin(\theta + c_3)]\, r^a \exp(b\theta) \tag{2.9}$$

$$\sigma_r = \operatorname{Re}\left\{Ar^a[(a + b^2 + 2)\chi + (1 + 2ib)\chi' - \chi'']\exp(b\theta) + \frac{\varphi' - \varphi''}{r^2}\right\} \tag{2.10}$$

$$\tau = -\operatorname{Re}\left\{Ar^a[(a + 1)b\chi + (b + i(a + 1))\chi' + i\chi'']\exp(b\theta) + \frac{i(\varphi' - \varphi'')}{r^2}\right\} \tag{2.11}$$

$$\sigma_\theta = \operatorname{Re}\left\{Ar^a[(a + 1)(a + 2)\chi + (2a + 3)\chi' + \chi'']\exp(b\theta) + \frac{\psi'' - \varphi'}{r^2}\right\} \tag{2.12}$$

Let us note that it can be seen by analogous means that the functions

$$E = \frac{E_0}{r^2}, \quad (\nu + 1)r^2 = c_1 + c_2 r\sin(\theta + c_3), \quad U = \bar{z}\chi(z) + \varphi(z) \tag{2.13}$$

satisfy (1.1). Here $z = \ln r - i\theta$ and $\chi(z), \varphi(z)$ are arbitrary analytic functions of the complex argument z. The function U in (2.13) has the same form as the known Muskhelishvili solution for a homogeneous body /6/.

3. **Examples.** Let us illustrate the method of determining χ and φ in (2.10)—(2.12) in the example of a radial stress distribution $(\tau = \sigma_\theta = 0)$. For this case we arrive, from (2.11), at three equalities

$$(a + 1)\chi + \chi' = 0, \quad (a + 1)\chi' + \chi'' = 0, \quad \varphi'' - \varphi' = 0$$

Integrating and then taking into account that $z = \ln r + i\theta$, we obtain

$$\chi = Ar^{-(a+1)}\exp[-i(a + 1)\theta], \quad \varphi = D_1 r\exp(i\theta) + D_2 \tag{3.1}$$

Here A, D_1, D_2 are arbitrary constants (complex, in the general case).

We find by substituting (3.1) into (2.10) and (2.12)

$$\sigma_\theta = 0, \ \sigma_r = \operatorname{Re}\left\{\frac{A}{r}[b^2 - a^2 - 2a - 2ib(a + 1)]\exp[b - i(a + 1)\theta]\right\} \tag{3.2}$$

$$A = A_1 + iA_2$$

Let us extract the real part in (3.2). We consequently obtain

$$\sigma_r = g(\theta)/r \tag{3.3}$$

$$g(\theta) = [a_1\cos(a + 1)\theta + a_2\sin(a + 1)\theta]\exp(b\theta)$$

$$a_1 = (b^2 - a^2 - 2a)A_1 + 2b(a + 1)A_2, \quad a_2 = (b^2 - a^2 - 2a)A_2 - 2b(a + 1)A_1 \tag{3.4}$$

The function (3.3) characterizes the radial distribution of the stress. Examples for such a stress are presented in /1,2,7/ and a method is indicated for determining the constants A_1 and A_2. Without going into details, let us say that, for instance, for the Vahart problem of a wedge (see /1/, Fig.25) these constants are found from the equilibrium conditions for the part of the wedge cut by an arc of arbitrary radius, which are determined by the formulas /2, 7/)

$$\int_{\theta_1}^{\theta_2} r\sigma_r\cos\theta d\theta = -p_1, \quad \int_{\theta_1}^{\theta_2} r\sigma_r\sin\theta d\theta = -p_2 \tag{3.5}$$

Here θ_1, θ_2 are angles measured from the $0x$ axis directed downward for a nonsymmetric location of the wedge relative to the $0x$ axis, and p_1, p_2 are components of the given pressure at the apex of the elastic wedge.

Taking account of (3.3), the integrals (3.5) are evaluated in elementary functions, and determination of the constants raises no difficulties. Problems on bending of a wedge /7,8/ and of a curved bar /2/ can be solved analogously.

Condition (1.4) for the Poisson's ratio ν can be satisfied by different methods be selecting the constants c_1, c_2, c_3, a, b in (1.3) and (2.6)

$$\nu + 1 = [c_1 + c_2 r\sin(\theta + c_3)]r^a\exp(b\theta) \tag{3.6}$$

For example, in the case of a truncated wedge $r_1 \leqslant r \leqslant r_2$, $\theta_1 \leqslant \theta \leqslant \theta_2$ let ν be given at three angular points on the boundary of this wedge $M_1(r_1, \theta_1; \nu_1)$, $M_2(r_2, \theta_2; \nu_2)$, $M_3(r_1, \theta_2; \nu_3)$. Inserting these values in (3.6) and solving the appropriate system of three equations for c_1, c_2, c_3, we obtain

$$c_1 = \frac{(\nu_3+1)\, r_2^{a+1} - (\nu_2+1)\, r_1^{a+1}}{(r_2 - r_1)\, r_1^{a} r_2^{a}} \exp(-b\theta_2)$$

$$\operatorname{tg} c_3 = \frac{B \sin\theta_2 - A \sin\theta_1}{A\cos\theta_1 - B\cos\theta_2}$$

$$A = \frac{\nu_3 + 1}{r_1^{a}} \exp(-b\theta_2) - c_1, \quad B = \frac{\nu_1 + 1}{r_1^{a}} \exp(-b\theta_1) - c_1$$

$$c_2 = \left[\frac{\nu_2 + 1}{r_2^{a}} \exp(-b\theta_2) - c_1\right] [r_2 \sin(\theta_2 + c_3)]^{-1}$$

For simplicity in the computation, assuming here $a = 1$, $b = 1$, $r_2 = 1$, $r_1 = 0.9$, $\nu_1 = 0.04$, $\nu_2 = 0.2$, $\nu_3 = 0{,}26$, $\theta_1 = -15°$, $\theta_2 = 30°$, we find $c_1 = 1.89$, $c_2 = -1.31$, $c_3 = 0{,}599$. The relief of the function $\nu(r, \theta)$ for this case is characterized by the Table 1, where values of $\nu \cdot 10^3$ are presented. Appropriate tables can be compiled for the Young's modulus E, and the shear modulus G by means of (1.3) and (2.6).

Table 1

r	$\theta = -15°$	$-10°$	$-5°$	$0°$	$5°$	$10°$	$15°$	$20°$	$25°$	$30°$
1.00	121	134	144	151	156	159	166	173	180	200
0.98	109	122	134	142	151	157	168	177	192	215
0.96	86	107	121	134	144	154	169	182	200	228
0.94	71	90	109	119	138	151	169	184	208	241
0.92	56	72	97	113	130	146	167	188	214	250
0.90	40	63	82	98	123	142	165	189	219	262

Let us note that if we put $c_1 = 0$ and $a = -1$ into (2.6) and (3.6), then ν will depend only on the angle θ, and the Young's modulus on r and θ. In this case the constants c_2 and c_3 are determined in terms of the given angles in the problem of a wedge (3.3), (3.4) for which the function $g(\theta)$ takes the simple form

$$g(\theta) = (b^2 - 3) A_1 \exp(b\theta)$$

Let us examine another approach. As before, let the function $\varphi(z)$ be determined by (3.1) and the function $\chi(z)$ in (2.10)— (2.12) will be sought in the form of a complex Fourier series

$$\chi(z) = \sum_{n=1}^{\infty} (A_n' + iA_n'') \exp(n\omega z) \qquad (3.7)$$

Here ω is a fixed characteristic number of the specific problem. The constants A_n', A_n'' should be determined from the obundary conditions.

Let us form the appropriate derivative of (3.7), and introduce them into (2.10)— (2.12) and then extract the real part. We consequently obtain

$$\sigma_r = \exp b\theta \sum_{n=1}^{\infty} r^{a+n\omega} [(\Delta_1 A_n' - 2bn\omega A_n'') \cos n\omega\theta - (\Delta_1 A_n'' + 2bn\omega A_n') \sin n\omega\theta] \qquad (3.8)$$

$$\tau = \exp b\theta \sum_{n=1}^{\infty} r^{a+n\omega} [(\Delta_2 A_n' - \Delta_3 A_n'') \cos n\omega\theta - (\Delta_2 A_n'' + \Delta_3 A_n') \sin n\omega\theta]$$

$$\sigma_\theta = \exp b\theta \sum_{n=1}^{\infty} \Delta_4 r^{a+n\omega} (A_n' \cos n\omega\theta - A_n'' \sin n\omega\theta)$$

$$\Delta_1 = a + b^2 + 2 + n\omega(1 - n\omega), \quad \Delta_2 = b(a + 1 + n\omega)$$

$$\Delta_3 = n\omega(a + 1 + n\omega), \quad \Delta_4 = (a+1)(a+2) + n\omega(2a + 3 + n\omega)$$

For $a + n\omega \neq -1$ these formulas can be used for the case when there are no stresses at the wedge apex $(r = 0)$. The formulas (3.8) contain $2n$ $(n = 1, 2, \ldots)$ arbitrary constants A_n', A_n'' for whose determination in the wedge domain $0 \leqslant r \leqslant R$, $\theta_1 \leqslant \theta \leqslant \theta_2$ the following problem can be formulated for example: $\sigma_r = F_1(\theta)$, $\tau = F_2(\theta)$ for $r = R$.

In this case the usual Fourier method must be used with the sole difference that the given functions $F_1(\theta)$ and $F_2(\theta)$ multiplied first by $\exp(-b\theta)$, must be expanded in the Fourier series in sines and cosines with argument $n\omega\theta$ in the range $\theta_1 \leqslant \theta \leqslant \theta_2$.

If the term $(B_n' + iB_n'')\exp(-n\omega z)$ is additionally inserted into (3.7), then the possibility is apparent of formulating analogous boundary conditions for a truncated cone.

In conclusion, we note that if it is required in (3.8) that $\sigma_\theta = \tau = 0$ (radial stress distribution), then the equality $\Delta_2 = 0$ will result in the condition $n\omega = -(a+1)$. Then $\Delta_3 = 0$, $\Delta_4 = 0$, $\Delta_1 = b^2 - a^2 - 2a$. Consequently, the signs of the sums in (3.7) and (3.8) must be omitted and we obtain the formulas (3.3) and (3.4) already examined.

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